A Method of Finding Integral Formulas which in certain Cases have a given ratio, where at the same time a method of summing continued fractions is presented *

Leonhard Euler

§1 As in recurring series each term is determined from one or more preceding terms according to a certain constant law, so I will consider series of such a kind here, in which each term is determined from one or more preceding ones according to a certain variable law. But since in such series the general formula expressing each term is not algebraic in most cases but transcendal, it will be convenient to exhibit each term by integral formulas; for them to yield definite values, I assume that after the integration a definite value is attributed to the variable quantity, such that each term results as a definite quantity; and now the principal question reduces to of what nature these integral formulas must be that each term is determined from one or more preceding ones according to a given law.

§2 To see this more clearly, consider the well-known series of these integral formulas

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$$\int \frac{dx}{\sqrt{1-xx}}, \quad \int \frac{xxdx}{\sqrt{1-xx}}, \quad \int \frac{x^4dx}{\sqrt{1-xx}}, \quad \int \frac{x^6dx}{\sqrt{1-xx}} \quad \text{etc.;}$$

if each one is integrated in such a way that it vanishes for $x=0$, but then the value 1 is attributed to the variable x, each term depends on the preceding one in such a way that

$$\begin{aligned}
\int \frac{xxdx}{\sqrt{1-xx}} &= \frac{1}{2}\int \frac{dx}{\sqrt{1-xx}}, \\
\int \frac{x^4dx}{\sqrt{1-xx}} &= \frac{3}{4}\int \frac{xxdx}{\sqrt{1-xx}}, \\
\int \frac{x^6dx}{\sqrt{1-xx}} &= \frac{5}{6}\int \frac{x^8dx}{\sqrt{1-xx}}
\end{aligned}$$

and in general

$$\int \frac{x^n dx}{\sqrt{1-xx}} = \frac{n-1}{n}\int \frac{x^{n-2}dx}{\sqrt{1-xx}}.$$

Hence it is plain that this general formula can be considered as general term of that series and each term results from the preceding one, if that one is multiplied by $\frac{n-1}{n}$.

§3 Hence similarly let us in general constitute a series of integral formulas

$$\int dv, \quad \int xdv, \quad \int xxdv, \quad \int x^3dv, \quad \int x^4dv \quad \text{etc.,}$$

such that the term corresponding to the index n is

$$\int x^{n-1}dv,$$

which integrals we want to assume to be taken in such a way that they vanish for $x=0$; but after the integration, let us attribute a certain constant value to the variable x, e.g., $x=1$ or any other number. Having constituted these things, the question reduces to how the function v of x must be chosen that each term is determined by one or two or more preceding terms according to a certain law; here one has to pay special attention to how many dimensions the index n rises in the propounded scale of relation; but in most cases in will not be necessary to go higher than the first dimension. Therefore, hence we will solve following problems.

PROBLEM 1

§4 *To find a function v that this relation among two subsequent terms holds*

$$\int x^n dv = \frac{\alpha n + a}{\beta n + b} \int x^{n-1} dv.$$

SOLUTION

Therefore, it is required here that

$$(\alpha n + a) \int x^{n-1} dv = (\beta n + b) \int x^n dv,$$

if a certain value is attributed to the variable x after the integration, of course. Therefore, since that condition can only hold, after that constant value was attributed to the variable x, let us put in general, while x is variable, that this equation holds

$$(\alpha n + a) \int x^{n-1} dv = (\beta n + b) \int x^n dv + V,$$

but the quantity V is of such a nature that it vanishes after that definite value has been assigned to the variable. Furthermore, since both integrals must be taken is such a way that they vanish for $x = 0$, it is necessary that also this quantity V vanishes in the same case.

§5 Since this equality must hold for all indices n, which we certainly always consider to be positive, it is easily understood that that quantity V must have the factor x^n; after this, that condition is already satisfied that for $x = 0$ also V becomes $= 0$. Therefore, let us set

$$V = x^n Q,$$

where Q denotes a suitable function of x, and which we at the same time desire to be of such a nature that it vanishes, if a certain value is attributed to x.

§6 Therefore, since it must be

$$(\alpha n + a) \int x^{n-1} dv = (\beta n + b) \int x^n dv + x^n Q,$$

differentiate this equation and, having divided the differential by x^{n-1}, one will get to this differential equation

$$(\alpha n + a)dv = (\beta n + b)xdv + nQdx + xdQ;$$

since this must hold for all values of n, the terms affected with that letter must cancel each other, whence we obtain these to equations

$$\text{I.} \quad (\alpha - \beta x)dv = Qdx \quad \text{and} \quad \text{II.} \quad (a - bx)dv = xdQ.$$

From the first $dv = \frac{Qdx}{\alpha - \beta x}$, from the other $dv = \frac{xdQ}{a-bx}$, which two values set equal to each other yield this equation $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a-bx}{\alpha - \beta x}$, which equation is resolved into these parts

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

whose integral will therefore be

$$\log Q = \frac{a}{\alpha} \log x - \frac{a\beta - b\alpha}{\alpha\beta} \log(\alpha - \beta x),$$

whence one deduces

$$Q = Cx^{\frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§7 From this value found for Q it is immediately clear that it it vanishes in the case $x = \frac{\alpha}{\beta}$, if just $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; but if this is not the case, then it is not clear how this quantity can vanish in any case. But having found this value Q, hence one will also find

$$dv = Cx^{\frac{a}{\alpha}} dx (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

and hence the term corresponding to the index n of our series will be

$$\int x^{n-1} dv = C \int x^{n + \frac{a}{\alpha} - 1} dx (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1},$$

but then it will be

$$V = Cx^{n + \frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

Here matters especially reduce to this that that quantity, aside from the case $x = 0$, additionally vanishes in another case.

COROLLARY 1

§8 Here, two cases occur, which require a peculiar expansion; the first, in which $\alpha = 0$; but then one has to start from the equation $\frac{dQ}{Q} = -\frac{(\alpha - bx)dx}{\beta xx}$, whence, by integration, one finds $\log Q = \frac{a}{\beta x} + \frac{b}{\beta}\log x$, and hence, taking e for the number whose hyperbolic logarithm is $= 1$, one concludes

$$Q = e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}},$$

which formula can only vanish if $\frac{a}{\beta x} = -\infty$ and hence $x = 0$, and so one would not have two cases in which $V = 0$, although nevertheless two are required. But aside from this it it will be

$$dv = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}} dx}{-\beta x}.$$

COROLLARY 2

§9 The other case requiring a peculiar expansion will be $\beta = 0$; but then it will be $\frac{dQ}{Q} = \frac{dx(a-bx)}{\alpha x}$, whence $\log Q = \frac{a}{\alpha}\log x - \frac{bx}{\alpha}$ and hence $Q = x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}}$, which formula vanishes in the case $x = \infty$, if just $\frac{b}{\alpha}$ was a positive number; but if $\frac{b}{\alpha}$ was a negative number, then Q vanishes in the case $x = -\infty$. Further, in this case it will be

$$dv = \frac{x^{\frac{a}{\alpha}} e^{\frac{-bx}{\alpha}} dx}{\alpha}.$$

SCHOLIUM

§9a Having observed these things in general, let us expand several special cases, in which we attribute certain values, which lead to already known cases, to the letters α, β and a, b.

EXAMPLE 1

§9b *Let integral formulas be in question that*

$$\int x^n dv = \frac{2n-1}{2n} \int x^{n-1} dv.$$

Therefore, since here it must be $(2n-1)\int x^{n-1}dv = 2n\int x^n dv$, in this case it will be $\alpha = 2$ and $a = -1$, but then $\beta = 2$ and $b = 0$; hence

$$\frac{dQ}{Q} = -\frac{dx}{2x(1-x)} = -\frac{dx}{2x} - \frac{dx}{2(1-x)},$$

thus, by integration

$$\log Q = -\frac{1}{2}\log x + \frac{1}{2}\log(1-x)$$

and hence

$$Q = C\sqrt{\frac{1-x}{x}}, \quad \text{therefore,} \quad V = Cx^n\sqrt{\frac{1-x}{x}}.$$

Further, since here $dv = \frac{Qdx}{2(1-x)}$, it will be

$$dv = \frac{Cdx\sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{Cdx}{2\sqrt{x-xx}};$$

therefore, having taken $C = 2$ it will be $dv = \frac{dx}{\sqrt{x-xx}}$ and our general formula

$$\int x^{n-1}dv = \int \frac{x^{n-1}dx}{\sqrt{x-xx}};$$

hence, since $V = x^n\sqrt{\frac{1-x}{x}}$, this quantity obviously vanishes for $x = 1$ such that our formula, if one sets $x = 1$ after the integration, meets the requirements. Therefore, if we put $x = yy$, that formula will take this form

$$2\int \frac{y^{2n-2}dy}{\sqrt{1-yy}},$$

which, having put $y = 1$ after the integration, yields this relation

$$\int \frac{y^{2n}dy}{\sqrt{1-yy}} = \frac{2n-1}{2n}\int \frac{y^{2n-2}dy}{\sqrt{1-yy}},$$

which contains the relations mentioned above (§ 2); for, hence it will be

$$\int \frac{yydy}{\sqrt{1-yy}} = \frac{1}{2}\int \frac{dy}{\sqrt{1-yy}},$$
$$\int \frac{y^4dy}{\sqrt{1-yy}} = \frac{3}{4}\int \frac{yydy}{\sqrt{1-yy}}$$

and

$$\int \frac{y^6dy}{\sqrt{1-yy}} = \frac{5}{6}\int \frac{y^4dy}{\sqrt{1-yy}}.$$

EXAMPLE 2

§10 *Let integral formulas be in question that*

$$\int x^n dv = \frac{\alpha n-1}{\alpha n}\int x^{n-1}dv.$$

Therefore, since here it must be $(\alpha n-1)\int x^{n-1}dv = \alpha n\int x^n dv$, in this case it will be $a=-1$, $\beta=\alpha$ and $b=0$, whence by the formulas given above one concludes

$$Q = Cx^{\frac{-1}{\alpha}}(\alpha-\alpha x)^{\frac{-a}{\alpha}} = Cx^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}},$$

which quantity obviously vanishes for $x=1$. But then it will be

$$dv = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}dx}{1-x},$$

whence our general formula will be

$$\int x^{n-1}dv = \int x^{x-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}dx = \int \frac{x^{n-\frac{1}{\alpha}-1}dx}{(1-x)^{1-\frac{1}{\alpha}}},$$

which can be simplified by setting $x=y^\alpha$; for, then it will take this form

$$\int \frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}},$$

where again after the integration one must set $y=1$. Hence it will be

$$\int \frac{y^{\alpha n+\alpha-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n-1}{\alpha n}\int \frac{y^{\alpha n-2}dy}{(1-y^\alpha)^{\frac{\alpha-1}{\alpha}}}$$

and hence the following special cases will result

$$\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}$$

and

$$\int \frac{y^{3\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}.$$

§11 Therefore, if one takes $\alpha = 1$ that it has to be

$$\int x^n dv = \frac{n-1}{n}\int x^{n-1}dv,$$

our general formula now, already expressed in y, will be $\int y^{n-2}dy$, whose value therefore is $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, whence the whole series of our integral formulas will go over into this one

$$\frac{1}{0}, \quad \frac{1}{1}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5}, \quad \frac{1}{6}, \quad \frac{1}{7} \quad \text{etc.}$$

§12 Let us also take $\alpha = \frac{1}{2}$ and now it will not necessary anymore to proceed to y. Therefore, in this case it will be

$$Q = \frac{(1-x)^2}{xx} \quad \text{and} \quad dv = \frac{(1-x)dx}{xx},$$

whence our general formula becomes

$$\int x^{n-1}dv = \int x^{n-3}(1-x)dx,$$

whose value expressed algebraically will therefore be

$$\frac{1}{n-2}x^{n-2} - \frac{1}{n-1}x^{n-1} = \frac{1}{(n-1)(n-2)},$$

whence the series of our formulas will become

$$\frac{1}{0\cdot(-1)}, \quad \frac{1}{0\cdot 1}, \quad \frac{1}{1\cdot 2}, \quad \frac{1}{2\cdot 3}, \quad \frac{1}{3\cdot 4}, \quad \frac{1}{4\cdot 5} \quad \text{etc.}$$

EXAMPLE 3

§13 *Let integral formulas be in question such that*

$$\int x^n dv = n \int x^{n-1} dv.$$

Therefore, since it must be $n \int x^{n-1} dv = 1 \int x^n dv$, it will be $\alpha = 1$, $a = 0$, $b = 1$, $\beta = 0$. Therefore, since $\beta = 0$, the case of Corollary 2 applies here and hence it will be $Q = e^{-x}$ and hence $V = e^{-x}x^n$, which quantity vanishes in the two cases $x = 0$ and $x = \infty$. Further, it will be $dv = e^{-x}dx$ and hence our general formula will become $\int x^{n-1}dxe^{-x}$, whence first the terms of the series will look as follows:

$$\int e^{-x}dx, \quad \int e^{-x}xdx, \quad \int e^{-x}xxdx, \quad \int e^{-x}x^3dx \quad \text{etc.},$$

having integrated which in such a way that they vanish for $x = 0$, then, having put $x = \infty$, the following rather simple series will result

$$1, \quad 1, \quad 1 \cdot 2, \quad 1 \cdot 2 \cdot 3, \quad 1 \cdot 2 \cdot 3 \cdot 4, \quad 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \quad \text{etc.},$$

which is the Wallisian hypergeometric series, whose general term hence is

$$\int x^{n-1}e^{-x}dx = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1).$$

§14 Therefore, by means of this general term it is possible to interpolate this series. So, if one wants to find the middle term between the first two, one must set $n = \frac{3}{2}$ and the value of this terms will be $\int e^{-x}dx\sqrt{x}$, whose value cannot be expressed algebraically by any means. But, using a singular method, I found that this term is equal to $\frac{1}{2}\sqrt{\pi}$, while π denotes the circumference of the circle whose diameter is $= 1$, whence here vice versa we see that $\int e^{-x}dx\sqrt{x} = \frac{\sqrt{\pi}}{2}$, having put $x = \infty$ after the integration, of course. But the term preceding this one corresponding to he index $\frac{1}{2}$ will be $= \sqrt{\pi}$, which is therefore equal to the formula $\int \frac{e^{-x}dx}{\sqrt{x}}$. Hence if we put $e^x = y$ here so that for $x = 0$ we have $y = 1$, but for $x = \infty$ we have $y = \infty$, then that integral formula $\int \frac{e^{-x}dx}{\sqrt{x}}$ goes over into this one $\int \frac{dy}{yy\sqrt{\log y}}$, which formula, if integrated in such a way that it vanishes for $y = 1$ but then one puts $y = \infty$, yields the value of $\sqrt{\pi}$. Further, if $y = \frac{1}{z}$, the limits of integration will be $z = 1$ and $z = 0$ and the integral formula will be

$$-\int \frac{dz}{\sqrt{-\log z}} \begin{bmatrix} \text{from} & z=1 \\ \text{to} & z=0 \end{bmatrix} = \sqrt{\pi}$$

or, having interchanged the limits of integration,

$$\int \frac{dz}{\sqrt{-\log z}} \begin{bmatrix} \text{from} & z=0 \\ \text{to} & z=1 \end{bmatrix} = \sqrt{\pi},$$

as I already noted some time ago.

Example 4

§15 *Let integral formulas be in question that*

$$\int x^n dv = \frac{1}{n}\int x^{n-1}dv \quad or \quad \int x^{n-1}dv = n\int x^n dv.$$

Here it is $\alpha = 0$ and $a = 1$, $\beta = 1$ and $b = 0$; therefore, this is the case treated in Corollary 1, whence it is concluded that it will be $Q = e^{\frac{1}{x}}$ and hence $V = x^n e^{\frac{1}{x}}$, which formula does not even vanish for $x = 0$, since the formula $e^{\frac{1}{0}}$ is equivalent to the infinity of an infinitesimal power. But here it miraculously happens that the case $x = -0$ renders the formula $e^{\frac{-1}{0}}$ vanishing. Of course, if ω denotes an infinitely small quantity, it will be $e^{\frac{1}{\omega}} = \infty^{\infty}$, but then on the other hand it will be $e^{\frac{-1}{\omega}} = \frac{1}{\infty^{\infty}} = 0$, whence we cannot exhibit a formula for our purpose here. One will certainly find $dv = -e^{\frac{1}{x}}\frac{dx}{x}$ so that our general formula is $-\int x^{n-2}dxe^{\frac{1}{x}}$, which is not useful for us.

§16 Therefore, if we put $\frac{1}{x} = y$, that integral formula goes over into this one $+\int \frac{e^y dy}{y^n}$. But now it will be $V = \frac{e^y}{y^n}$, which formula vanishes for $y = -\infty$. But no matter how we transform this expression, the same inconvenience will always occur. But this case can be resolved as follows. For, let the first term of the series we are looking for be $= \omega$, from which according to the prescribed rule the subsequent terms proceed this way

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4} & \cdots & \frac{\omega}{1\cdot 2\cdot 3\cdots(n-1)}. \end{array}$$

But above we saw that the value of this formula $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n-1)$ is expressed by this integral $\int x^{n-1} e^{-x} dx$ having extended the integration from $x = 0$ to $x = \infty$; therefore, it is just necessary that the transfer this integral formula to the denominator, and the general term of the series we are trying to find will be

$$\frac{1}{\int x^{n-1} e^{-x} dx},$$

whence is clearly seen that the task can not be solved by a simple integral formulas, what is also to be noted on the other cases, in which the quantity V can not vanish in two cases; for, then it is just necessary to invert the fraction $\frac{\alpha n + a}{\beta n + b}$ and to transfer the integral formula to the denominator.

SCHOLIUM

§17 If not $\alpha = 0$ or $\beta = 0$, which cases we already covered, the resolution of our problem can always be reduced to the case in which both letters α and β are equal to one. For, since it must be

$$\int x^n dv = \frac{\alpha n + a}{\beta n + b} \int x^{n-1} dv,$$

put $x = \frac{\alpha y}{\beta}$ and it will be

$$\frac{\alpha}{\beta} \int y^n dv = \frac{\alpha n + a}{\beta n + b} \int y^{n-1} dv,$$

which equation is reduced to this form

$$\int y^n dv = \frac{n + a : \alpha}{n + b : \beta} \int y^{n-1} dv.$$

If we now write a instead of $\frac{a}{\alpha}$ and b instead of $\frac{b}{\beta}$, this formula must be resolved

$$\int y^n dv = \frac{n+a}{n+b} \int y^{n-1} dv,$$

whose resolution, if we write y instead of x and one instead of α and β, from the above solution first yields

$$Q = Cy^a (1-y)^{b-a},$$

which therefore vanishes for $y = 1$, if just $b > a$; but then the formula itself will be

$$\int y^{n-1} dv = C \int y^{n+a-1} dy (1-y)^{b-a-1};$$

but if $b < a$, this solution, as we saw, cannot hold; but in this case one has to assume this form $\frac{1}{\int y^{n-1} dv}$ for the term of our series, such that then it must be

$$\frac{1}{\int y^n dv} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} dv}$$

or

$$\int y^n dv = \frac{n+b}{n+a} \int y^{n-1} dv,$$

whose resolution having permuted the letters a and b yields

$$Q = C y^b (1-y)^{a-b},$$

which now vanishes in the case $y = 1$, if $a > b$; and then the general formula will be

$$\int y^{n-1} dv = C \int y^{n+b-1} dy (1-y)^{a-b-1}.$$

Therefore, whether it is $b > a$ or $a > b$, the solution has no difficulty anymore.

§18 But if it was either $\alpha = 0$ or $\beta = 0$, one can write 1 instead of the other; hence, if it must be

$$\int x^n dv = \frac{n+a}{b} \int x^{n-1} dv,$$

because of $\alpha = 1$ and $\beta = 0$ our general solution gives

$$\frac{dQ}{Q} = \frac{dx}{x}(a - bx),$$

whence one concludes $Q = C x^a e^{-bx}$, which formula vanishes for $x = \infty$, if just b was a positive; but then the general term becomes

$$\int x^{n-1} dv = C \int x^{n+a-1} dx e^{-bx}.$$

But on the other hand the number b cannot be negative, since otherwise the prescribed condition would not be met.

§19 Let us also consider the other case in which $\alpha = 0$ and $\beta = 1$ and hence the prescribed condition

$$\int x^n dv = \frac{a}{n+b} \int x^{n-1} dv,$$

whence

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a - bx).$$

But hence for Q a value would result which, aside from the case $x = 0$, could not vanish; therefore, the general formula must be set $\frac{1}{\int x^{n-1} dv}$ such that it must be

$$\int x^n dv = \frac{n+b}{a} \int x^{n-1} dv,$$

whence

$$\frac{dQ}{Q} = \frac{dx}{x}(b - ax) \quad \text{and hence} \quad Q = Ce^{-ax} x^b,$$

which expression vanishes for $x = \infty$, since a must necessarily be a positive number; but then it will be

$$dv = Ce^{-ax} x^b dx,$$

whence the general formula of the series will be

$$\frac{1}{C \int x^{n+b-1} dx e^{-ax}}.$$

PROBLEM 2

§20 *Let T denote the term corresponding to the index n in the series which we intend to consider, but let T' denote the following term and let this condition be propounded to be satisfied*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')} T.$$

SOLUTION

Since here two values occur, this condition is satisfied in the most convenient manner, if the general term T is considered as the product of two factors. Therefore, set $T = RS$ and let the following term be $= R'S'$ and find the formulas R and S such that

$$R' = \frac{\alpha n + a}{\beta n + b}R \quad \text{and} \quad S' = \frac{\alpha' n + a'}{\beta' n + b}S;$$

for, then taking $T = RS$ the prescribed condition will obviously be satisfied. Therefore, this way one will find formulas of the kind $\int x^{n-1}dv$ or inverse ones for R and S, what suffices for the general solution, whence we want to illustrate this in an example.

EXAMPLE

§21 *Let the general formula T be in question such that*

$$T' = \frac{nn - cc}{nn}T.$$

Therefore, let us resolve T into two factors R and S and set

$$R' = \frac{n-c}{n}R \quad \text{and} \quad S' = \frac{n+c}{n}S.$$

If we set $R = \int x^{n-1}dv$ for the first formula, from the general solution, where it will be $\alpha = 1$, $a = -c$, $\beta = 1$ and $b = 0$, it will be

$$Q = Cx^{-c}(1-x)^c,$$

which form obviously vanishes for $x = 1$; and hence, since

$$V = Cx^{n-c}(1-x)^c,$$

this form also vanishes in the case $x = 0$, if just $n > c$, which can be assumed without worries, since we assumed the exponent n to grow to infinity and mostly just fractions are taken for c. Therefore, hence it will be

$$R = C\int x^{n-c-1}(1-x)^{c-1}dx.$$

§22 Hence one could already deduce the other value of the letter S, writing just $-c$ instead of c, but then it would not further be $Q=0$ for $x=1$, whence one has to assume the inverse formula $\frac{1}{\int x^{n-1}dv}$ for the formula S, that it has to be

$$\int x^n dv = \frac{n}{n+c}\int x^{n-1}dv;$$

since here $\alpha=1$, $a=0$, $\beta=1$ and $b=c$, one finds

$$Q=C(1-x)^c,$$

which form is obviously $=0$ for $x=1$; but hence it results

$$dv=C(1-x)^{c-1}dx,$$

therefore, we will have

$$S=\frac{1}{C\int x^{n-1}(1-x)^{c-1}dx};$$

consequently, our general formula in question will be

$$T=\frac{\int x^{n-c-1}(1-x)^{c-1}dx}{\int x^{n-1}(1-x)^{c-1}dx}.$$

§23 Therefore, if we put the first term of our series proceeding through factors $=A$, the series itself will be

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A, & \frac{1-cc}{1}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}A, & \frac{1-cc}{1}\cdot\frac{4-cc}{4}\cdot\frac{9-cc}{9}A \quad \text{etc.} \end{array}$$

hence, if we take $c=\frac{1}{2}$, the series will be

$$A, \quad \frac{1\cdot 3}{2\cdot 3}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}A, \quad \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}A \quad \text{etc.};$$

whose term corresponding to the index n therefore is

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}dx}{\int x^{n-1}(1-x)^{-\frac{1}{2}}dx},$$

which for $x = yy$ goes over into this form

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}dy}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}dy},$$

whence it is plain that the first term will be

$$A = \int \frac{dy}{\sqrt{1-yy}} : \int \frac{ydy}{\sqrt{1-yy}} = \frac{\pi}{2},$$

having put $y = 1$ after the integration, of course.

PROBLEM 3

§24 *Let T denote the term corresponding to index n of the series and let T' and T'' be the following terms for the indices $n+1$ and $n+2$; if among three subsequent terms such a relation is propounded that*

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'',$$

to investigate the formula for T, by which the general term of this series is expressed.

SOLUTION

Assume the integral formula $\int x^{n-1}dv$ for T and take the integral in such a way that it vanishes for $x = 0$, and the following term will be $T' = \int x^n dv$ and $T'' = \int x^{n+1}dv$, if a definite value is attributed to the variable x after the integration, of course. But as long as this quantity x is considered as a variable, let us put that

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'' + x^n Q$$

and it is perspicuous that Q must be a function of x of such a kind which vanishes, of that definite value is substituted for x, which must be different from zero, since we already assumed that all these formulas vanish for $x = 0$. But if after the calculation this condition cannot be satisfied by any means, this will be an indication that our problem cannot be solved this way, i.e. that its general term T is exhibited by such a simple differential formula $\int x^{n-1}dv$.

§25 Now let us differentiate the equation just constituted and, having done the division by x^{n-1}, the following equation will result

$$(\alpha n + a)dv = (\beta n + b)xdv + (\gamma n + c)xxdv + nQdx + xdQ,$$

which, since the terms affected by the letter n must cancel each other, is split into the following two equations

$$1. \quad \alpha dv = \beta xdv + \gamma xxdv + Qdx,$$

$$2. \quad adv = bxdv + cxxdv + xdQ,$$

from the first of which

$$dv = \frac{Qdx}{\alpha - \beta x - \gamma xx},$$

from the other

$$dv = \frac{xdQ}{a - bx - cxx},$$

the second of which values divided by the first yields

$$\frac{dQ}{Q} = \frac{dx(a - bx - cxx)}{x(\alpha - \beta x - \gamma xx)},$$

from which integration the value of Q must be found, having done which it will be seen clearly whether it can vanish in a certain case, aside from $x = 0$. But here it is especially to be noted, if this integral involves a factor of the kind $e^{\frac{\lambda}{x}}$, that then the solution will also not succeed, since for $x = 0$ that factor will only involve a power of infinity, that, even though it is multiplied by x^n, the product still remains infinite.

§26 Therefore, if those prescribed conditions could be satisfied, then, having found the value of the letter Q, which we put to become $= 0$ for $x = f$, one will have

$$dv = \frac{Qdx}{\alpha - \beta x - \gamma xx}$$

and the general formula containing the nature of the series will be

$$T = \int x^{n-1} dv = \int \frac{x^{n-1} Q dx}{\alpha - \beta x - \gamma xx},$$

whose integral extended from $x = 0$ to $x = f$ will yield the value of the term T corresponding to the index n.

SCHOLIUM

§27 But having found such a relation among three subsequent terms of a series, one can in usual manner form a continued fraction, whose value can be assigned. For, if the characters

$$T', \quad T'', \quad T''', \quad T'''' \quad \text{etc.}$$

denote the terms following after T to infinity, from the relations among them the following formulas will be deduced. From the relation

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T''$$

one deduces

$$(\alpha n + a)\frac{T}{T'} = \beta n + b + \frac{(\gamma n + c)(\alpha n + \alpha + a)}{(\alpha n + \alpha + a)T' : T''}.$$

From the following relation

$$(\alpha n + \alpha + a)T' = (\beta n + \beta + b)T'' + (\gamma n + \gamma + c)T'''$$

one deduces

$$(\alpha n + \alpha + a)\frac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a)T'' : T'''}.$$

In like manner, the following relations will give

$$\begin{aligned}(\alpha n + 2\alpha + a)\frac{T''}{T'''} &= \beta n + 2\beta + b\frac{(\gamma n + 2\gamma + c)(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a)T''' : T''''},\\ (\alpha n + 3\alpha + a)\frac{T'''}{T''''} &= \beta n + 3\beta + b\frac{(\gamma n + 3\gamma + c)(\alpha n + 4\alpha + a)}{(\alpha n + 4\alpha + a)T'''' : T'''''};\end{aligned}$$

hence it is manifest, if in the first formula one continuously substitutes the following values in order, that a continued fraction will result, whose value will be equal to the formula $(\alpha n + a)\frac{T}{T'}$.

§28 Therefore, if we write the numbers 1, 2, 3, 4 etc. instead of n, we will be able to solve the following problems on continued fractions.

PROBLEM 4

Having propounded a continued fraction of this form

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}},$$

to assign its value.

SOLUTION

Consider that relation among three subsequent quantities T, T', T'' in general, which is

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

and from the preceding problem find the value of T, if it is possible, of course, expressed in this way

$$T=\int x^{n-1}dv=\int\frac{x^{n-1}Qdx}{\alpha-\beta x-\gamma xx},$$

whose integral is to be extended from $x=0$ to $x=f$; having found this form, put

$$\int\frac{Qdx}{\alpha-\beta x-\gamma xx}=A \quad \text{and} \quad \int\frac{xQdx}{\alpha-\beta x-\gamma xx}=B,$$

such that A and B are the values of T for the cases $n=1$ and $n=2$; having determined them, by the preceding results the value of the propounded continued fraction will be $=\frac{(\alpha+a)A}{B}$. Therefore, let us apply this investigation to the following examples.

Example 1

§29 *To investigate the value of the famous continued fraction once given by Brouncker for the quadrature of the circle, which reads*

$$2+\cfrac{1\cdot 1}{2+\cfrac{3\cdot 3}{2+\cfrac{5\cdot 5}{2+\text{etc.}}}}$$

Since all integer parts are constant and $=2$, for our general formula it will be

$$\beta+b=2, \quad 2\beta+b=2, \quad 3\beta+b=2 \quad \text{etc.};$$

therefore, it will be $\beta=0$ and $b=2$; but for the numerators of the following simple fractions, since they consist of two factors, for the first factors it will be

$$\gamma+c=1, \quad 2\gamma+c=3, \quad 3\gamma+c=5, \quad 4\gamma+c=7 \quad \text{etc.},$$

whence one concludes $\gamma=2$ and $c=-1$, for the others on the other hand it will be

$$2\alpha+a=1, \quad 3\alpha+a=3, \quad 4\alpha+a=5 \quad \text{etc.},$$

whence $\alpha=2$ and $a=-3$. But from these values we conclude this equation

$$\frac{dQ}{Q}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

which, having cancelled $1+x$, yields

$$\frac{dQ}{Q}=-\frac{dx(3-x)}{2x(1-x)},$$

whence by integration

$$\log Q=-\frac{3}{2}\log x+\log(1-x) \quad \text{and hence} \quad Q=\frac{1-x}{x^{\frac{3}{2}}},$$

from which value it further follows

$$A = \int \frac{(1-x)dx}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{dx}{2x(1+x)\sqrt{x}},$$

$$B = \int \frac{(1-x)dx}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{dx}{2(1+x)\sqrt{x}}.$$

§30 But in these formula one detects that inconvenience that that first integral cannot be rendered vanishing for $x = 0$. But this inconvenience can easily be removed, if we truncate the continued fraction by its first term and find the value of this continued fraction

$$2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + \text{etc.};}}$$

if it was found to be $= s$, the value of the propounded fraction will be $= b + \frac{1}{s}$. But now, after having made the comparison, as before we have $\beta = 0$ and $b = 2$, but then $\gamma = 2$ and $c = +1$, $\alpha = 2$ and $a = -1$, whence it follows

$$\frac{dQ}{Q} = -\frac{dx(1+2x+xx)}{2x(1-xx)} = -\frac{dx(1+x)}{2x(1-x)},$$

whence by integration

$$\log Q = -\frac{1}{2}\log x + \log(1-x) \quad \text{and hence} \quad Q = \frac{1-x}{\sqrt{x}},$$

from which value we will now have

$$A = \int \frac{(1-x)dx}{2(1-xx)\sqrt{x}} = \frac{1}{2}\int \frac{dx}{(1+x)\sqrt{x}}$$

and

$$B = \frac{1}{2}\int \frac{dx\sqrt{x}}{1+x};$$

here, since $Q = \frac{1-x}{\sqrt{x}}$, its value obviously vanishes for $x = 1$, whence those integrals are to be extended from the limit $x = 0$ to the limit $x = 1$.

§31 Now to find these integrals more easily, let us set $x = zz$, such that the limits of integrations are still $z = 0$ and $z = 1$, and it will be

$$A = \int \frac{dz}{1+zz} = \arctan z = \frac{\pi}{4}$$

and

$$B = \int \frac{zzdz}{1+zz} = 1 - \frac{\pi}{4}$$

and so we will have $s = \frac{\pi}{4-\pi}$, whence the value of the Brounckerian continued fraction is $1 + \frac{4}{\pi}$, precisely as Brouncker had already found it.

EXAMPLE 2

§31a *To investigate the value of this generalised Brounckerian continued fraction*

$$b + \cfrac{1\cdot 1}{b + \cfrac{3\cdot 3}{b + \cfrac{5\cdot 5}{b + \text{etc.}}}}$$

To avoid the above inconvenience, let us omit the first term and find

$$s = b + \cfrac{3\cdot 3}{b + \cfrac{5\cdot 5}{b + \text{etc.},}}$$

since then the value in question will be $= b + \frac{1}{s}$. Therefore, it will now be $\beta = 0$ and $b = b$, $\gamma = 2$, $c = 1$, $\alpha 2$ and $a = -1$, whence

$$\frac{dQ}{Q} = -\frac{dx(1+bx+xx)}{2x(1-xx)}$$

and thus,

$$\log Q = -\frac{1}{2}\log x - \frac{b-2}{4}\log(1+x) + \frac{b+2}{4}\log(1-x)$$

and hence

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

which formula obviously becomes $= 0$ by putting $x = 1$, if $b + 2$ was a positive number, of course, whence

$$dv = \frac{(1-x)^{\frac{b-2}{4}} dx}{2(1+x)^{\frac{b+2}{4}} \sqrt{x}}.$$

But hence one will conclude

$$A = \frac{1}{2} \int \frac{(1-x)^{\frac{b-2}{4}} dx}{(1+x)^{\frac{b+2}{4}}} \quad \text{and} \quad B = \frac{1}{2} \int \frac{(1-x)^{\frac{b-2}{4}} dx \sqrt{x}}{(1+x)^{\frac{b+2}{4}}}$$

or, putting $x = zz$, we will have

$$A = \int \frac{(1-zz)^{\frac{b-2}{4}} dz}{(1+zz)^{\frac{b+2}{4}}} \quad \text{and} \quad B = \int \frac{(1-zz)^{\frac{b-2}{4}} zzdz}{(1+zz)^{\frac{b+2}{4}}},$$

which both integrals are to be extended from $z = 0$ to $z = 1$. But from these values A and B it will be $s = \frac{A}{B}$; therefore, the value of the propounded continued fraction will be $= b + \frac{1}{s} = b + \frac{B}{A}$.

§32 Therefore, if one puts $b = 2$ here, the case explained before depending on the quadrature of the circle results, in which case the integral formula becomes even rational. But whenever the exponents $\frac{b-2}{4}$ and $\frac{b+2}{4}$ are not integer numbers, then the letters A and B can be expressed neither in terms of circular arcs nor in terms of logarithms. As if was $b = 4$, it will be

$$A = \int \frac{dz\sqrt{1-zz}}{(1+zz)^{\frac{3}{2}}},$$

whose value could be exhibited via elliptical arcs. But if b was an odd number, these values become a lot more transcendental, such that we have to be content with these letters A and B. But otherwise, if those exponents become integer numbers, the whole task can be completed by circular arcs.

§33 But those exponents $\frac{b-2}{4}$ and $\frac{b+2}{4}$ will be integer numbers, if b was a number of the form

$$b = 4i + 2;$$

for, then it will be

$$A = \int \frac{(1-zz)^i dz}{(1+zz)^{i+1}} \quad \text{and} \quad B = \int \frac{(1-zz)^i zzdz}{(1+zz)^{i+1}};$$

therefore, it will be worth one's while to teach how these cases must be expanded, since already Wallis contemplated them.

§34 Since here this whole task reduces to the reduction of integral formulas of this kind to simpler ones, let us in general consider the form $P = \frac{z^m}{(1+zz)^n}$, whose differential can be exhibited in the following form

$$1. \quad dP = \frac{mz^{m-1}dz}{(1+zz)^n} - \frac{2nz^{m+1}dz}{(1+zz)^{n+1}},$$

$$2. \quad dP = \frac{mz^{m-1}dz}{(1+zz)^{n+1}} - \frac{(2n-m)z^{m+1}dz}{(1+zz)^{n+1}},$$

$$3. \quad dP = -\frac{(2n-m)z^{m-1}dz}{(1+zz)^n} + \frac{2nz^{m-1}dz}{(1+zz)^{n+1}},$$

whence we deduce these three reductions of integrals

$$\text{I.} \quad \int \frac{z^{m+1}dz}{(1+zz)^{n+1}} = \frac{m}{2n}\int \frac{z^{m-1}dz}{(1+zz)^n} - \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n},$$

$$\text{II.} \quad \int \frac{z^{m+1}dz}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int \frac{z^{m-1}dz}{(1+zz)^{n+1}} - \frac{1}{2n-m} \cdot \frac{z^m}{(1+zz)^n},$$

$$\text{III.} \quad \int \frac{z^{m-1}dz}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int \frac{z^{m-1}dz}{(1+zz)^n} + \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n},$$

by means of which reductions the task can be completed in the cases $b = 4i+2$ and can be reduced to the formula $\frac{\pi}{4}$, if one takes $z = 1$ after the integration, of course.

§35 Let $i = 1$ and hence $b = 6$ and it will be

$$A = \int \frac{(1-zz)dz}{(1+zz)^2} \quad \text{and} \quad B = \int \frac{(1-zz)zzdz}{(1+zz)^2}.$$

Therefore, by the third reduction we will now find

$$\int \frac{dz}{(1+zz)^2} = \frac{1}{2}\int \frac{dz}{1+zz} + \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

and by the first reduction

$$\int \frac{zzdz}{(1+zz)^2} = \frac{1}{2}\int \frac{dz}{1+zz} - \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4},$$

further,

$$\int \frac{z^4dz}{(1+zz)^2} = \frac{3}{2}\int \frac{zzdz}{1+zz} - \frac{1}{2}\cdot\frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

From these values one now concludes $A = \frac{1}{2}$ and $B = \frac{\pi}{2} - \frac{3}{2}$ and hence $\frac{B}{A} = \pi - 3$, whence these summation will arise

$$3+\pi = 6 + \cfrac{1\cdot 1}{6 + \cfrac{3\cdot 3}{6 + \cfrac{5\cdot 5}{6 + \cfrac{7\cdot 7}{6 + \text{etc.}}}}}$$

§36 Now let $i = 2$ and $b = 10$ and it will be

$$A = \int \frac{(1-zz)^2dz}{(1+zz)^3} \quad \text{and} \quad B = \int \frac{zz(1-zz)^2dz}{(1+zz)^3}.$$

To investigate the values of these integrals, let us expand the following formulas

$$\int \frac{dz}{(1+zz)^3} = \frac{3}{4}\int \frac{dz}{(1+zz)^2} + \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4},$$

$$\int \frac{zzdz}{(1+zz)^3} = \frac{1}{4}\int \frac{dz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{\pi}{32},$$

$$\int \frac{z^4dz}{(1+zz)^3} = \frac{3}{4}\int \frac{zzdz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4},$$

$$\int \frac{z^6dz}{(1+zz)^3} = \frac{5}{4}\int \frac{z^4dz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

From these values one now deduces $A = \frac{\pi}{8}$ and $B = 2 - \frac{5\pi}{8}$ and hence $\frac{B}{A} = \frac{16-5\pi}{\pi}$, whence the following summation emerges

$$\frac{5\pi+16}{\pi} = 10 + \cfrac{1\cdot 1}{10 + \cfrac{3\cdot 3}{10 + \cfrac{5\cdot 5}{10 + \text{etc.}}}}$$

§37 If b would be a negative number, the investigation would not bear any further difficulty. For, if in general

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-b + \cfrac{\gamma}{-c + \cfrac{\delta}{-e + \text{etc.}}}}},$$

it will always be

$$-s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{etc.}}}}},$$

whence, if one has the value of this expression, the same taken negatively will give the value of that one.

Example 3

§38 *Let this continued fraction, whose value is to be investigated, be propounded*

$$1 + \cfrac{1\cdot 1}{3 + \cfrac{3\cdot 3}{5 + \cfrac{5\cdot 5}{7 + \cfrac{7\cdot 7}{9 + \text{etc.}}}}}$$

To apply the continued fractions mentioned above [§ 28], having omitted the first term, let

$$s = 3 + \cfrac{3\cdot 3}{5 + \cfrac{5\cdot 5}{7 + \cfrac{7\cdot 7}{9 + \text{etc.}}}}$$

and it will be $\beta + b = 3$, $2\beta + b = 5$ and hence $\beta = 2$ and $b = 1$, then, as before, $\alpha = 2$, $a = -1$, $\gamma = 2$ and $c = +1$; but having found s the value in question will be $= 1 + \frac{1}{s}$. Therefore, we will now have

$$\frac{dQ}{Q} = -\frac{dx(1+x+xx)}{2x(1-x-xx)}.$$

But on the other hand

$$\frac{1+x+xx}{x(1-x-xx)} = \frac{1}{x} + \frac{2+2x}{1-x-xx},$$

whence

$$\log Q = -\frac{1}{2}\log x - \int \frac{dx(1+x)}{1-x-xx}.$$

Further, for find the formula $\int \frac{dx(1+x)}{1-x-xx}$, let us set the denominator

$$1 - x - xx = (1-fx)(1-gx)$$

and it will be $f + g = 1$ and $fg = -1$, whence

$$f = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad g = \frac{1-\sqrt{5}}{2}.$$

Now set

$$\frac{1+x}{1-x-xx} = \frac{\mathfrak{A}}{1-fx} + \frac{\mathfrak{B}}{1-gx},$$

whence one will find

$$\mathfrak{A} = \frac{1+f}{f-g} \quad \text{and} \quad \mathfrak{B} = -\frac{1+g}{f-g},$$

or, having substituted the values given above for f and g, it will be

$$\mathfrak{A} = \frac{\sqrt{5}+3}{2\sqrt{5}} \quad \text{and} \quad \mathfrak{B} = \frac{\sqrt{5}-3}{2\sqrt{5}},$$

having found which it will be

$$\int \frac{dx(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f}\log(1-fx) - \frac{\mathfrak{B}}{g}\log(1-gx)$$
$$= -\frac{1+\sqrt{5}}{2\sqrt{5}}\log(1-fx) - \frac{\sqrt{5}-1}{2\sqrt{5}}\log(1-gx),$$

whence it will be

$$\log Q = -\frac{1}{2}\log x + \frac{\sqrt{5}+1}{2\sqrt{5}}\log(1-fx) + \frac{\sqrt{5}-1}{2\sqrt{5}}\log(1-gx),$$

as a logical consequence

$$Q = \frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

which value vanish in two case, first for

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2},$$

second for

$$x = \frac{1}{g} = -\frac{1+\sqrt{5}}{2};$$

but no matter which one we use, matters reduce to the same.

§39 But from this value we will have

$$A = \int \frac{Qdx}{1-x-xx} \quad \text{and} \quad B = \int \frac{Qxdx}{1-x-xx},$$

whence one further deduces

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B};$$

hence the sum of the propounded continued fraction will be $1+\frac{B}{A}$. But hence nothing more can be concluded because of the non only irrational but even, because of the surdic exponents, transcendental formulas.

EXAMPLE 4

§40 *Let this continued fraction be propounded*

$$b+\cfrac{1\cdot 1}{b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\cfrac{4\cdot 4}{b+\text{etc.}}}}}$$

where $\beta=0$, $b=0$.

Now let us consider this form

$$s=b+\cfrac{2\cdot 2}{b+\cfrac{3\cdot 3}{b+\text{etc.}}}$$

having found which the value in question will be $=b+\frac{1}{s}$. Therefore, we will have $\gamma+c=2$, $2\gamma+c=3$ and hence $\gamma=1$ and hence $c=1$, further it will be $\alpha=\gamma=1$, $a=0$ and $c=1$. Therefore, we hence calculate

$$\frac{dQ}{Q}=-\frac{dx(bx+xx)}{x(1-xx)}=-\frac{dx(b+x)}{1-xx}$$

and hence

$$\log Q=-\frac{b}{2}\log\frac{1+x}{1-x}+\frac{1}{2}\log(1-xx)$$

and thus,

$$Q=\frac{(1-x)^{\frac{b}{2}}\sqrt{1-xx}}{(1+x)^{\frac{b}{2}}}=\frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

which quantity obviously vanishes for $x=1$. Therefore, hence it will be

$$A=\int\frac{Qdx}{1-xx}=\int\frac{(1-x)^{\frac{b+1}{2}}dx}{(1+x)^{\frac{b-1}{2}}(1-xx)}=\int\frac{(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}}$$

and

$$B = \int \frac{x(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}},$$

but then it will be $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$ and hence the sum in question $b + \frac{B}{A}$.

§41 Let us now go through the principal cases and first let $b = 1$ and it will be

$$A = \int \frac{dx}{1+x} = \log(1+x) = \log 2 \quad \text{and} \quad B = \int \frac{xdx}{1+x} = x - \int \frac{dx}{1+x} = 1 - \log 2$$

and hence $b + \frac{B}{A} = \frac{1}{\log 2}$; therefore, hence this summation will arise

$$\frac{1}{\log 2} = 1 + \cfrac{1 \cdot 1}{1 + \cfrac{2 \cdot 2}{1 + \cfrac{3 \cdot 3}{1 + \text{etc.}}}}$$

§42 Now let $b = 2$ and it will be

$$A = \int \frac{dx\sqrt{1-x}}{(1+x)^{\frac{3}{2}}} \quad \text{and} \quad B = \int \frac{xdx\sqrt{1-x}}{(1+x)^{\frac{3}{2}}}.$$

To render these formulas rational, let us set

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z$$

and it will be $x = \frac{1-zz}{1+zz}$, whence $z = 1$ and $z = 0$ correspond to the limits of integration $x = 0$ and $x = 1$; but then it will be

$$1 + x = \frac{2}{1+zz} \quad \text{and} \quad dx = -\frac{4zdz}{(1+zz)^2}$$

and hence one concludes

$$A = -2\int \frac{zzdz}{1+zz} = -2z + 2\arctan z = 2 - \frac{\pi}{2},$$

further,

$$B = -2\int \frac{zzdz}{(1+zz)^2} + 2\int \frac{z^4dz}{(1+zz)^2}.$$

Therefore, by the reductions shown above (§ 35), if we permute the limits of integration here, of course, that we have

$$B = +\int \frac{zzdz}{(1+zz)^2} - 2\int \frac{z^4dz}{(1+zz)^2},$$

it will be

$$B = 2\left(\frac{\pi}{8} - \frac{1}{4}\right) - 2\left(\frac{5}{4} - \frac{3\pi}{8}\right) = \pi - 3,$$

whence this summation follows

$$\frac{2}{4-\pi} = 2 + \cfrac{1\cdot 1}{2 + \cfrac{2\cdot 2}{2 + \cfrac{3\cdot 3}{2 + \cfrac{4\cdot 4}{2 + \text{etc.}}}}}$$

which is as simple as the Brounckerian continued fraction.

§43 If we put $b = 0$, the continued fraction goes over into the following continuous product

$$\frac{1\cdot 1}{2\cdot 2}\cdot\frac{3\cdot 3}{4\cdot 4}\cdot\frac{5\cdot 5}{6\cdot 6}\cdot\frac{7\cdot 7}{8\cdot 8}\cdot\text{etc.};$$

but in this case

$$A = \int \frac{dx}{\sqrt{1-xx}} \quad \text{and} \quad B = \int \frac{xdx}{\sqrt{1-xx}} = 1,$$

whence the value of this product is deduced to be $\frac{2}{\pi}$, which agrees extraordinarily with already known results, since this product is the Wallisian progression, of course.

EXAMPLE 5

§44 *Let this continued fraction be propounded, where $\beta = 0$, $b = b$ and the numerators are the triangular numbers,*

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Having omitted the first term let us set

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

and first represent the numerations by products this way

$$3 = 2 \cdot \frac{3}{2}, \quad 6 = 3 \cdot \frac{4}{2}, \quad 10 = 4 \cdot \frac{5}{2},$$

the first of which must be compared with the formulas $\gamma + c$, $2\gamma + c$, $3\gamma + c$, the latter on the other hand must be compared with the formulas $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, and it will be $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, whence it will be

$$\frac{dQ}{Q} = \frac{dx\left(\frac{1}{2} - bx - xx\right)}{x\left(\frac{1}{2} - xx\right)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

or

$$\frac{dQ}{Q} = \frac{dx}{x} - \frac{2bdx}{1 - 2xx},$$

whose integral is

$$\log Q = \log x - \frac{b}{\sqrt{2}} \log \frac{1 + x\sqrt{2}}{1 - x\sqrt{2}},$$

therefore,

$$Q = \frac{x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

which formula vanishes in the case $x = \frac{1}{\sqrt{2}}$. Therefore, hence it will be

$$dv = \frac{2x(1-x\sqrt{2})^{\frac{b}{\sqrt{2}}}dx}{(1-2xx)(1+x\sqrt{2})^{\frac{b}{\sqrt{2}}}}.$$

Let $\frac{b}{\sqrt{2}} = \lambda$ and it will be

$$A = 2\int \frac{x(1-x\sqrt{2})^{\lambda}dx}{(1-2xx)(1+x\sqrt{2})^{\lambda}} = 2\int \frac{x(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}}$$

and

$$B = 2\int \frac{xx(1-x\sqrt{2})^{\lambda-1}dx}{(1+x\sqrt{2})^{\lambda+1}},$$

having put $x = \frac{1}{\sqrt{2}}$ after the integration; but then $s = \frac{A}{B}$ and hence the value of the propounded fraction is $= b + \frac{B}{A}$.

§45 Therefore, these values can only be assigned in a convenient manner, if $\lambda = \frac{b}{\sqrt{2}}$ was a rational number. Therefore, let $b = \sqrt{2}$ or $\lambda = 1$ and it will be

$$A = 2\int \frac{xdx}{(1+x\sqrt{2})^2} \quad \text{and} \quad B = 2\int \frac{xxdx}{(1+x\sqrt{2})^2}.$$

Hence by integration one concludes

$$A = \log(1+x\sqrt{2}) - \frac{x\sqrt{2}}{1+x\sqrt{2}}$$

and hence for $x\sqrt{2} = 1$ it will be $A = \log 2 - \frac{1}{2}$; but then one finds

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2}\cdot\log 2,$$

whence, because of $b = \sqrt{2}$ it will be $b + \frac{B}{A} = \frac{1}{\sqrt{2}(2\log 2 - 1)}$, whence this summation follows

$$\frac{1}{\sqrt{2}(2\log 2 - 1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

§46 But the continued fractions, to which we are mostly led by numerical calculations, usually have a form of this kind

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

where all numerators are 1, the denominators a, b, c, d, e etc. on the other hand are integer numbers. But by means of our method one easily finds the value of such formulas, even if the numbers a, b, c, d, e constitute an arithmetic progression, what we want to show in the following example.

Example

§47 *Let this continued fraction be propounded*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b+\cfrac{1}{5\beta+b+\text{etc.}}}}}$$

where $\alpha=0$, $\beta=0$, $a=1$, $c=1$.

Hence

$$\frac{dQ}{Q}=-\frac{dx(1-bx-xx)}{\beta xx},$$

whence

$$\log Q=\frac{1}{\beta x}+\frac{b}{\beta}\log x+\frac{x}{\beta}\quad\text{and}\quad Q=e^{\frac{1+xx}{\beta x}}x^{\frac{b}{\beta}},$$

which expression cannot vanish in any case, even though it is multiplied by x^n, if β was a positive number, of course. But if negative numbers are taken for β, say $\beta=-m$, then the value $Q=x^{\frac{-b}{m}}e^{\frac{-(1+xx)}{mx}}$ obviously vanishes, so for $x=0$ as for $x=\infty$. But hence

$$dv = \frac{x^{\frac{-b}{m}} e^{\frac{-1-xx}{mx}} dx}{mxx},$$

whence we will have

$$A = \frac{1}{m} \int \frac{dx}{x^{2+\frac{b}{m}} e^{\frac{1+xx}{mx}}} \quad \text{and} \quad B = \frac{1}{m} \int \frac{dx}{x^{1+\frac{b}{m}} e^{\frac{1+xx}{mx}}}.$$

Having found these values the formula will express the sum of this continued fraction

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

whence that formula taken negatively, i.e. $-\frac{A}{B}$, will express the value of this continued fraction

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

which could therefore be assigned, if just the integral formulas A and B could by found and be extended from the limit $x=0$ to $x=\infty$. But these formulas are of such a nature that their integration cannot be expressed in terms of familiar quantities by any means, which is still no obstruction that the fraction $\frac{A}{B}$ can involve sufficiently known values, even though we cannot assign them by any means yet.

§48 But concerning such formulas, I discovered the following two, whose values can be exhibited conveniently:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

and

$$n - \cfrac{1}{3n - \cfrac{1}{5n - \cfrac{1}{7n - \cfrac{1}{9n - \text{etc.}}}}} = \cot\frac{1}{n}$$

The first of these continued fractions compared to the formulas of the last example yields $m - b = n$, $2m - b = 3n$ and hence $m = 2n$ and $b = n$, whence

$$A = \frac{1}{2n}\int \frac{dx}{x^{\frac{5}{2}}e^{\frac{1+xx}{2nx}}} \quad \text{and} \quad B = \frac{1}{2n}\int \frac{dx}{x^{\frac{3}{2}}e^{\frac{1+xx}{2nx}}},$$

whence we already learn, if these two formulas are integrated from the limit $x = 0$ to the limit $x = \infty$, that it will then be

$$\frac{A}{B} = \frac{1 + e^{\frac{2}{n}}}{1 - e^{\frac{2}{n}}},$$

although there is no other analytical way to prove this agreement.